Hodge Conjecture

Let X now be a complex manifold, and $T_{\mathbb{C}}X = T_{\mathbb{C}}^{1,0}X \oplus T_{\mathbb{C}}^{0,1}X$ the complexified $2n$-Real tangent bundle. This gives a splitting $\Omega^k_{\mathbb{C}}X = \bigoplus_{p+q=k} \Omega^{p,q}X$ (as a C^∞-bundle). So considering the de Rham resolution of $\mathbb{C}_X$:

$$0 \to \mathbb{C}_X \to \Omega^0 \to \Omega^1 \to \cdots, \quad \Omega^k \text{ - sheaf of smooth } k\text{-forms, (fine, hence acyclic).}$$

Thus $H^\bullet(X, \mathbb{C}_X) \cong H^\bullet(\Gamma(X, \Omega^\bullet))$, which is de Rham's theorem. Using the splitting $\Omega^k = \oplus \Omega^{p,q}$, we get a bicomplex

$$\left.\begin{array}{ccccccc}
0 & \to & \Omega^2_{hol} & \to & \cdots & & \cdots \\
 & & \partial\uparrow & & \uparrow & & \uparrow \\
0 & \to & \Omega^1_{hol} & \to & \Omega^{1,0} & \to & \Omega^{1,1} \\
 & & \partial\uparrow & & \uparrow & & \uparrow \\
0 & \to & \Omega^0_{hol} & \xrightarrow{\bar\partial} & \Omega^{0,0} & \xrightarrow{\bar\partial} & \Omega^{0,1} \\
 & & \uparrow & & & & \\
 & & 0 & & & &
\end{array}\right\}$$

Hodge-de Rham Spectral Sequence: $H^q(X, \Omega^p_{hol}) \Rightarrow H^{p+q}(X, \mathbb{C}_X)$.
If X is Kähler + compact:

$$H^k_{dR}(X, \mathbb{C}_X) = \bigoplus_{p+q=k} H^{p,q}(X)$$

This is the complex hodge decomposition.

If X is projective algebraic, then one can prove the degeneration of the Hodge-de Rham spectral sequence algebraically by reducing to characteristic p! (Deligne-Illusie). Read this?

We also have Poincaré Duality: $H^k_{dR}(X) \times H^{2n-k}_{dR}(X) \longrightarrow \mathbb{C}$, $(\omega, \eta) \mapsto \int_X \omega \wedge \eta$ (again, X is compact Kähler). This descends to a finer duality:

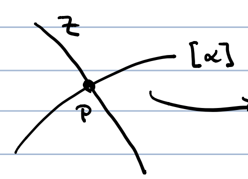
$$H^{p,q}(X) \otimes H^{p',q'}(X) \longrightarrow 0$$

unless $p' = n-p$, $q' = n-q$ $\Rightarrow$ $H^{p,q} = (H^{n-p,n-q})^*$. So this gives reflection about $y = -x$ in the Hodge diamond.

If $Z \subset X$ is a closed subvariety of X of dimension k, we would like Z to define a functional on Ω^{2k} via $\omega \mapsto \int_Z \omega \in \mathbb{C}$. Hopefully this would descend to cohomology, and by Poincare duality an actual cohomology class $[Z]$. This however has issues. For various reasons (see Griffith's-Harris), singularities can be delt with, so we can get that all such Z give a cohomology class, so we get a cycle map $cl_X : A_k(X) \longrightarrow H^{2n-2k}_{dR}(X)$, where $A_k(X)$ is the k^{th} Chow group, graded by dimension. Moreover, one can show $cl_X(Z) \in H^{n-k,n-k}(X)$. The cocycles in the image of cl_X are called algebraic classes (or analytic classes).

Example: $k = n-1$
Z is then a divisor, so $cl_X(Z) = c_1(\mathcal{O}_X(Z)) \in H^{1,1}(X)$.

Now every $\alpha \in H_{2n-2k}(X)$ can be represented by a cycle $[\alpha]$ which intersects Z transversally.



$\longrightarrow$ $i_p(Z, [\alpha])$ - the intersection multiplicity at p. $\Big\}$ $\Rightarrow (Z \cdot [\alpha]) = \sum_{p \in [\alpha] \cap Z} i_p([\alpha], Z) \in \mathbb{Z}$.

One can show (with hard work), this gives a functional $(-\cdot Z): H_{2n-2k}(X) \longrightarrow \mathbb{Z}$ (or w/ $\mathbb{Q}$ coefficients....). Now $H_{2n-2k}(X, \mathbb{Q})^* = H^{2n-2k}(X, \mathbb{Q}) \longleftrightarrow H^{2n-2k}_{dR}(X)$ via the cycle map. So we could define $H^s_{dR}(X, \mathbb{Q}) = \operatorname{Im}(H^s(X, \mathbb{Q}) \to H^s_{dR}(X, \mathbb{C}))$.

<u>Corollary:</u> $cl_X(Z) \in H^{2n-2k}_{dR}(X) \cap H^s_{dR}(X, \mathbb{Q}) = H^{2n-k, 2n-k}(X, \mathbb{Q})$.

<u>Hodge Conjecture:</u> The $\mathbb{Q}$-vector space $H^{p,p}(X, \mathbb{Q})$ is spanned by $cl_X(Z)$ for subvarieties Z of X.

<u>Theorem (Lefschetz):</u> The Hodge conjecture is true for $p=1$, i.e., everything in $H^{1,1}$ is the chern class of a divisor.

Equivariant Sheaves (Again)

Let X be a top space and $G \overset{\sigma}{\curvearrowright} X$ a top. group, with $F \in Sh(X)$. We say F is G-equivariant if given $\Theta: \sigma^* F \xrightarrow{\sim} \pi^* F$, for:

$$G \times G \times X \rightrightarrows G \times X \underset{\pi}{\overset{\sigma}{\rightrightarrows}} X,$$

Θ satisfies the relevant cocycle conditions. Hence a G-equivariant sheaf is a pair $(F, \Theta) \in Sh_G(X) \xrightarrow[\text{forget}]{} Sh(X)$.

<u>Ex:</u> Take G discrete, then $\Theta \longleftrightarrow$ lifting G-action from X to F. This means given $g \in G$, $X \xrightarrow{g} X$, get an iso: $\Theta_g: g^* F \xrightarrow{\sim} F$.

Claim: If $F \in Sh_G(X)$, then $H^i(X, F)$ is a G-module. The difficulty is that we may not have a G-equivariant resolution of injectives! One way out is Čech methods, but we use the canonical flabby resolution, due to Godement. This resolution naturally inherits the G-equivariant structure.

There is also a Godement resolution of Étale sheaves. Now suppose $G \curvearrowright Y$ and $f: Y \to X$ is a G-map, where X has the trivial G-action (every orbit in Y lies in a fiber of f). If we have $\mathcal{F} \in Sh(X)$, then we claim $f^* \mathcal{F} \in Sh_G(Y)$. Indeed we have $f^* \mathcal{F}(U) = f^* \mathcal{F}(gU)$, so define our isomorphism to be this identification.

<u>Prop:</u> Suppose $f: Y \to X$ is a principal homogeneous G-space. Then $f^*: Sh(X) \longrightarrow Sh_G(Y)$ is an equivalence. (Also applies to étale sheaves).

<u>Main example:</u> X/k, k-field, $\overline{X} = X \times \operatorname{Spec} k_s$. Now $Gal(k_s/k) \curvearrowright k_s$. Then if $f: \overline{X} \to X$, $f^* \mathcal{F}$ is a G_k-equivariant sheaf on $\overline{X}$, hence $H^\bullet(\overline{X}_{et}, f^* \mathcal{F})$ is a G_k-module.

<u>Def:</u> $H^\bullet(X, \mathbb{Z}_\ell(1)) = \varprojlim H^\bullet(X, \mu_{\ell^n})$, where ℓ is a prime with $(\ell, \operatorname{char} k) = 1$. Called the Tate twist.

Worth checking how this relates to $\mathbb{Z}_\ell$-cohomology.

If G is an abelian group, we have an inverse system of torsion subgroups $\cdots \longrightarrow G_{\ell^2} \longrightarrow G_\ell$, and the inverse limit is the Tate module $T_\ell(G)$. (Interesting example: take G to be an elliptic curve).

Def: $H^\cdot(X, \mathbb{Z}_\ell(m)) = H^\cdot(X, \mathbb{Z}_\ell(1)) \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(1)^{\otimes m}$

Thm: Let X/k, $\overline{X} = X \times k_s$, $f: \overline{X} \to X$ the projection, $F \in Sh(X_{et})$ a torsion sheaf. Then for $k \subset k_s \subset K$, and $X' = \overline{X} \times K \longrightarrow \overline{X} \xrightarrow{f} X$ with composition g, then g^*F is $Gal(k_s/k)$ equivariant.

Now let $X/\mathbb{C}$ be a smooth proper scheme. Hence there is $k = \mathbb{Q}(a_1, \dots, a_\ell) \subset \mathbb{C}$, with X_0/k, s.t. $X = X_0 \times_k \mathbb{C}$ (this realization is important when reducing to characteristic p). So $H^\cdot(X, \mathbb{Z}/n\mathbb{Z})$ has a $Gal(\mathbb{C}/k)$ action, which factors through $Gal(\overline{k}/k)$. But we have a natural isomorphism $H^\cdot(X_{et}, \mathbb{Z}/n\mathbb{Z}) \cong H^\cdot_{sing}(X(\mathbb{C}), \mathbb{Z}/n\mathbb{Z})$, which then inherits the $Gal(\overline{k}/k)$ action.

We will eventually define the cycle map in Étale cohomology, by sending $Z \subset X$ to a cohomology class $cl_X(Z) \in H^{2k}(X, \mu_n^{\otimes c})$, for $(n, \text{char } k) = 1$.

Take X smooth over $k = \overline{k}$, and a prime ℓ with $(\ell, \text{char } k) = 1$. Then we have a projective system of étale sheaves: $\cdots \longrightarrow \mu_{\ell^3}^{\otimes c} \xrightarrow{(-)^\ell} \mu_{\ell^2}^{\otimes c} \xrightarrow{(-)^\ell} \mu_\ell^{\otimes c} \longrightarrow 1$, $c > 0$. Hence:

$$\begin{array}{ccccc} & \cdots \longrightarrow & H^{2c}(X, \mu_{\ell^2}^{\otimes c}) & \longrightarrow & H^{2c}(X, \mu_\ell^{\otimes c}) \\ & & \Psi & & \Psi \\ \underset{\text{codim } c}{Z \subset X} & & cl_X(Z) & \longrightarrow & cl_X(Z) \end{array}$$

So we get a class in $H^{2c}(X, \mathbb{Z}_\ell(c)) = \varprojlim H^{2c}(X, \mu_{\ell^n}^{\otimes c})$, and so in $H^{2c}(X, \mathbb{Q}_\ell(c))$. We want to show $A^c(X) \xrightarrow{cl_X} H^{2c}(X, \mathbb{Z}_\ell(c))$ is a ring homomorphism. Let X be defined on a subfield $k_0 \subset k$: $X = X_0 \times_{k_0} \operatorname{Spec} k \circlearrowleft Gal(k/k_0)$, hence a Galois action on $H^{2c}(X, \mathbb{Q}_\ell(c))$.

If Z is an irreducible subvariety of X defined over $k_0 \subset k' \subset k$, then Z is $Gal(k/k')$ invariant, hence its image in $H^{2c}(X, \mathbb{Q}_\ell(c))$ is $Gal(k/k')$-invariant (as the cycle map is equivariant with the Galois group).

Corollary: The subspace of algebraic classes in $H^{2c}(X, \mathbb{Q}_\ell(c))$ consists of classes ξ s.t. ξ is fixed by a subgroup $Gal(k/k') \subset Gal(k/k_0)$ for some f.g. extension k'/k_0.

Tate Conjecture: The converse to the above.

Note we have yet to define the cycle map. We aim to define a homomorphism of graded rings:

$$\bigoplus_{c=0}^{\dim X} A^c(X) \longrightarrow \bigoplus_{c=0}^{\dim X} H^{2c}(X_{et}, \mu_{\ell^n}^{\otimes c}), \quad (n, \text{char } k) = 1$$

Case 1: $c = 1$. Start with a divisor class $D \subset X$. It defines a line bundle, hence a class $[D]$ in $H^1(X_{et}, \mathbb{G}_m)$. Now we have the Kummer sequence w/ a map $d: H^1(X, \mathbb{G}_m) \to H^2(X, \mu_n)$ so set $d[D] = cl_X(D)$.

Case 2: Z is a scheme-theoretic intersection $D_1 \cap \dots \cap D_c$. Use the cup product on cohomology:

$$\begin{array}{ccc} H^2(X, \mu_n) \otimes \cdots \otimes H^2(X, \mu_n) & \longrightarrow & H^{2c}(X, \mu_n^{\otimes c}) \\ cl_X(D_1) \ \cdots \ cl_X(D_c) & \longrightarrow & cl_X(Z). \end{array}$$

Of course, one needs some work to show this goes from the Chow ring.

For the general case, need cohomology with compact support. $H^p_Z(X,F)$, the p^{th} cohomology of X in F w/ support in Z is by definition $H^p(Z, Ri^!F)$, where $i: Z \hookrightarrow X$ is the inclusion. Note if $F = \mathbb{Z}$, $H^p_Z(X,F) = H^p(X, X\setminus Z; \mathbb{Z})$, the relative cohomology.

<u>Exercise 29:</u> Consider a triple of inclusions $V \underset{\text{open}}{\subset} U \underset{\text{open}}{\subset} X$, get a long-exact sequence
$\cdots \to H^p_{X\setminus V}(X,F) \to H^p_{X\setminus U}(X,F) \to H^p_{U\setminus V}(X,F) \to H^{p+1}_{X\setminus V}(X,F) \to \cdots$ $F \in Sh(X)$.

In the étale setting cohomology with compact supports still works, and we also have purity:

<u>Thm:</u> Let $Z \overset{i}{\hookrightarrow} X/k$ be a smooth sub-var. of codim c, $(n, \operatorname{char} k) = 1$. F be a locally constant n-torsion sheaf. Then $\mathcal{H}^p(Ri^!F) = 0$ if $p \neq 2c$, locally isomorphic to i^*F.

<u>Thm:</u> In the same setup $(k = \bar{k})$, with $F = \mu_n^{\otimes c}$. Then there is a natural isomorphism $\mathcal{H}^{2c}(Ri^!\mu_n^{\otimes c}) = \underline{\mathbb{Z}/n\mathbb{Z}}$ - constant sheaf on Z.

<u>Def:</u> Given $Z \subset X$ smooth, the image of 1 in $\Gamma(Z, \mathbb{Z}/n\mathbb{Z}) \cong \Gamma(Z, Ri^!\mu_n^{\otimes c}) \cong H^{2c}_Z(X, \mu_n^{\otimes c})$ is the fundamental class of Z, S_Z. Thus the cycle map is given by composing with $H^{2c}_Z(X,\mu_n^{\otimes c}) \longrightarrow H^{2c}(X, \mu_n^{\otimes c})$.

<u>Example:</u> Construction of fundamental class if $Z = D$ - a divisor.
We have:

$$0 \to i_*Ri^!\mathbb{G}_m \longrightarrow \mathbb{G}_m \longrightarrow Rj_*j^*\mathbb{G}_m \longrightarrow 0, \qquad D \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U$$

and taking cohomology:

$$\begin{array}{ccccccccc} H^0(U,\mathbb{G}_m) & \to & H^1_Z(X,\mathbb{G}_m) & \to & H^1(X,\mathbb{G}_m) & \to & H^1(U,\mathbb{G}_m) & \to & \cdots \\ \| & & \| & & \| & & \| & & \\ \Gamma(U, \mathcal{O}_U^*) & \longrightarrow & \mathbb{Z} & \longrightarrow & \operatorname{Pic} X & \longrightarrow & \operatorname{Pic} U & & \\ f & \longmapsto & \operatorname{ord}_D(f) & & & & & & \end{array}$$

restriction of line bundle.

Now taking the Kummer sequence vertically in the exact triangle above:

$$\begin{array}{ccccc} H^1_Z(X,\mathbb{G}_m) & \xrightarrow{n} & H^1_Z(X,\mathbb{G}_m) & \longrightarrow & H^2_Z(X,\mu_n) \\ & & 1 & \longmapsto & S_{Z/X}. \end{array}$$

Now how to define $cl_X(Z)$ or $S_{Z/X}$ if Z is singular?

<u>Lemma:</u> Let $Z \subset X$ be a closed reduced subscheme of codimension r. Then $H^s_Z(X, \mu_n^{\otimes r}) = 0$ for $s < 2r$.

<u>Proof:</u> Descending induction on r. If $r = \dim X$, $Z \subset X$ is a collection of points $\Rightarrow$ smooth pair. This follows easily.

$r+1 \Rightarrow r$: Take $X - Z^{sing} = U$. Then $U \cap Z$ is smooth, dense in Z, and $X - U = Z^{sing}$ has codimension at least $r+1$. Now the above exercise on the triple $X \supset U \supset X - Z$, we get a long exact sequence, from which we conclude the result. ∎

<u>Digression:</u> Coherent Cohomology
Let X be regular, $Z \subset X$ be a subscheme. If $\mathcal{F} \in Coh(X)$, can define the local cohomology $H^{\cdot}_Z(X, F) = Ext^{\cdot}_{Coh(X)}(i_*\mathcal{O}_Z, \mathcal{F})$. Consider then $\mathcal{F} = \mathcal{O}_X$. Then we claim $H^i_Z(X, \mathcal{O}_X) = 0$ if $i <$ codim Z. This is related to depth (Cohen-Macaulay).

Now taking $s = 2r$, we get an isomorphism $H^{2r}_Z(X, \mu_n^{\otimes r}) \xrightarrow{\sim} H^{2r}_{U \cap Z}(U, \mu_n^{\otimes r})$.
This gives us the cycle map. Recall it is a ring hom:

$$\begin{tikzcd} Y \arrow[d, "\text{flat}"] \\ X \arrow[d, "\text{proper}"] \\ Z \end{tikzcd} \qquad \begin{tikzcd} \bigoplus_{c=0}^{\dim X} A^c(X) \arrow[r] \arrow[d] & \bigoplus_{c=0}^{\dim X} H^{2c}(X, \mu_n^{\otimes r}) \arrow[d] \\ \bigoplus_{c=0}^{\dim Y} A^c(Y) \arrow[r] & \bigoplus_{c=0}^{\dim Y} H^{2c}(Y, \mu_n^{\otimes r}) \end{tikzcd}, \qquad (n, \operatorname{char} k) = 1.$$

Have a pushforward $\oplus A^c(X) \to \oplus A^c(Z)$.